Jones. Prof. W. W. Beman informs me that Jones's exposition of logarithms, as given in his *Synopsis palmariorum matheseos*, 1706, was based on Halley's treatment of 1695, but a posthumous paper by Jones, published in the *Philosophical Transactions* for the year 1771, gives the new definition. Whether Jones printed this definition before Gardiner is still undetermined. The one whose influence was greatest in emphasizing the new view was Euler, who, in his *Introductio*, 1748, Chap. VI, §102, gives the definition involving exponents. In this same chapter Euler gives an exposition of negative and fractional exponents and calls attention to the multiple values of a number having a fractional exponent, an explanation seldom found in mathematical treatises of that time. That the new definition of a logarithm was in every way a step in advance has been doubted by some writers. Certain it is that it involves internal difficulties of a serious nature.

[*To be continued.*]

MINIMUM COURSES IN ENGINEERING MATHEMATICS.

By S. EPSTEEN, University of Colorado.

Introduction.—This paper was suggested by a number of inquiries as to the nature and content of the course in engineering mathematics at the University of Colorado. This course is based on three entrance units in Mathematics, and consists of algebra, trigonometry, analytic geometry, the calculus, and least squares as prescribed courses and differential equations, higher calculus, vector analysis, Fourier's series and other advanced courses as electives.

The following outline[1] is not that of a complete course in engineering mathematics, nor even the average course, but, as the title of the paper indicates, the minimum course. The average course is based on this minimum but contains more material, both theoretical and applied. This outline gives an irreducible minimum. A course falling below this standard may be a good trade school course, it may be a most useful and practical course in many respects, but it is not a course in engineering mathematics.

Mathematics and Engineering.—Engineering mathematics is in no sense trade school mathematics or practical arithmetic. A trade school may have little use for mathematics as a science, but the engineering college demands a knowledge of principles as well as facts. This is particularly noticeable in the recent advances in the profession of civil engineering which have been along the lines laid down by Rankine and not by Trautwine. To the engineering student mathematics is as essential as anatomy is to the surgeon, as chemistry is to the apothecary, as drill is to the army officer. The professor of engineering is certainly on firm ground when he takes the stand that the mathematics taught to his students should not be too abstract on the one hand nor too concrete on the other. If the subject matter is too abstract it is unintelligible or uninteresting to the beginner; if it is

[1] The Editors, while in sympathy with the broad purposes of this paper, share no responsibility for the details of the suggested programs.

too concrete the science degenerates to the mere performing of certain mechanical operations—to a common tool instead of a valuable instrument.

Mathematicians and Engineers.—Until very recently there was a complete lack of understanding and sympathy between mathematicians and engineers. In the proceedings of the Society for the Promotion of Engineering Education, volume 17, page 39, the chairman of the joint committee makes the following statement: "The mathematicians expected that the engineers would tell them that what they wanted is practical arithmetic, while the engineers fully expected that the mathematicians would tell them that what they wanted is 'pure mathematics,' the queen of the sciences.' But as the session progressed everyone was astonished to find that no conflict was precipitated." The situation has greatly improved since that time. In volume 19 of the same proceedings we find a complete syllabus of a minimum course in mathematics for engineering students. The report is by an individual, not by the entire committee. While the author's eminence as a mathematician is above all question, his experience with engineering is not as extensive as his experience in pure mathematics. As might be expected under such circumstances, his report is substantially sound, but it needs revision in details. For instance, in the algebra course, no mention is made of graphic algebra. This topic should be included. On the other hand, harmonic progressions are mentioned, with the comment that they are not of great importance. This topic should be excluded.

The Engineering Student and the Arts Student.—The purpose and aim of the prospective engineer are so radically different from that of the general student that the content of the course should be different from that which is generally given to the student who elects mathematics as a part of a general education. Experience proves that the best instruction to engineering students is not given by instructors who have no patience with anything except pure mathematics, but by those who have had some training in engineering or at least considerable training in mathematical physics. That is to say, the most sympathetic instructors are generally those who are near enough to engineers to take some interest in the concrete problems themselves as distinct from their solutions.

The Freshman and Sophomore Years.

Entrance Requirements.—Many high schools are rebelling against college domination. Their stand is well taken. Only a small percentage of their students go to college and it is therefore the obvious duty of the high school to concentrate its chief efforts on the majority. Most engineering colleges demand three units in mathematics for entrance, consisting of one and one-half years of algebra, one year of plane geometry and one-half year of solid geometry. Under the existing circumstances this is all that can be effectively enforced, at least in some parts of the country, even if more is desirable, and even though more is frequently offered.

Engineering Algebra.—While the amount of time devoted to each course differs considerably in the various colleges, the general average for algebra is

three times per week for one semester (or the equivalent); that is, 54 class periods less holidays and quiz days, making a total of 45 class periods. In view of the fact that the course must necessarily begin with a brief review, it is perfectly obvious that the ordinary text book contains far too much material. Many authors argue that this is desirable in that it gives the instructor a wider range of selection and it impresses upon the student the fact that when he has finished the course he has not learned all there is to know about it. Whatever merit or demerit there may be in the second argument, the first is not strictly sound. Even an experienced instructor does not always know what to select from the immense amount of material at his disposal, while the inexperienced instructor is quite at sea. By inexperienced is meant not merely those who have done little or no teaching, but also those whose interests are exclusively in pure mathematics and who have had no practical experience with engineers, their points of view and their standards of value. For example, even experienced instructors frequently select theory of equations as a suitable topic, whereas, in reality, this chapter is relatively unimportant to an engineer as compared with topics that are omitted when this one is included. An actual inquiry shows that the professors of electrical engineering and mechanical engineering place almost no value on this topic; and the professors of civil engineering state that while they occasionally want the students to know how to solve a cubic or a quartic equation, the need is so rare that they would willingly forego this topic for others.

The minimum course in algebra should consist of the following chapters: (1) Review of fractional and negative exponents, reduction of surds, imaginaries, zero, infinity; (2) Logarithms; (3) Variables, functions, graphs; (4) Simultaneous linear equations, determinants of the second and third orders, graphical solution; (5) Quadratic equations; (6) Simultaneous quadratics; (7) Variation; (8) Arithmetic and geometric progressions (number of terms finite, infinite); (9) Binomial theorem, the series for e and e^x; (10) Complex numbers, vectors, addition, subtraction, multiplication and division; (11) Partial fractions; (12) Permutations, combinations, probability. There are about forty formulas in these chapters which should be understood and memorized so that they may become the student's everyday working tools—his mental machinery. There should be numerous applications to simple problems taken from actual engineering practice.

Engineering Trigonometry.—In trigonometry the problem of a suitable course is much simpler than in algebra. The general average of time allowed to this course is two hours per week for one semester or the equivalent; that is, 36 days less holidays and quiz days, making about 30 class periods. Trigonometry is needed primarily for three specific purposes—surveying, physics, calculus. The minimum course can therefore be summarized in about forty formulas under the following headings: (1) Trigonometric functions; (2) Right triangles; (3) The addition theorems; (4) Inverse trigonometric functions, trigonometric equations; (5) The oblique triangle; (6) Circular measure, graphical representation. If the students have had logarithms early in the algebra course, so that this topic may be omitted in the trigonometry, it is possible to

get in a short chapter on the right spherical triangle. The students of civil engineering need spherical trigonometry in some of their other courses. A brief study of the right spherical triangle is not enough, but it constitutes an introduction, and this is all that can be reasonably expected under the time limitations.

Engineering Analytics.—Most colleges devote five hours per week for one semester to analytic geometry; that is, about 80 actual class periods. While one cannot cover this subject fully in that time, it is sufficient for a fair introduction. Indeed, if eighty periods could be given to algebra also, instead of forty-five, the results would be large in proportion to the extra time. Analytic geometry is an indispensable part of any good mathematical curriculum. Although highly interesting in itself, it would be a still better course if frequent applications were introduced. The newer books on algebra, trigonometry and calculus abound in interesting and important applications, but not so in analytic geometry. Authors of the text books on this subject still follow slavishly in the footsteps of their predecessors and concrete applications are exceedingly rare in a book on analytics.

The minimum course should consist of about seventy-five formulas under the following headings: (1) The straight line; (2) The circle; (3) The parabola; (4) The ellipse; (5) The hyperbola; (6) Polar coordinates; (7) Transformation of coordinates; (8) Higher plane curves; (9) Coordinates in space. It is well to take up a chapter on the general equation of the second degree if time permits, but this is by no means an essential part of the minimum course, as the author of the syllabus (Art. 5) thinks. Indeed, a chapter on higher plane curves, including the cycloid, hypocycloid, epicycloid, involute and others, is far more important.

Engineering Calculus.—Calculus is doubtless the most valuable and effective instrument the engineer can possess. Of all the subjects in the curriculum no other is so far reaching and powerful in its applications. Still, one sometimes hears a successful engineer say "I use algebra and trigonometry a great deal but never the calculus." Whenever a man makes this assertion, it usually indicates that he had defective instruction in the calculus when he first took the course. Naturally, he could not use an instrument which he did not understand. He may have been a good engineer without the calculus, but he would have been a better engineer with it.

A minimum course in the calculus should consist of a thorough understanding of the proofs and applications of about one hundred formulas and theorems included in the following chapters: (1) Functions, continuity, limits (a very brief chapter); (2) Differentiation; (3) Simple applications to geometry, physics, mechanics, maxima and minima; (4) Integration as anti-differentiation; (5) Integration as the limit of a sum, definite integrals; (6) Applications of integration; (7) Centers of mass, moments of inertia, radii of gyration (special emphasis on plane sections); (8) Series, indeterminate forms; (9) Approximate methods of integration; (10) Differential equations with applications. All the important formulas and theorems should be memorized.

THE JUNIOR YEAR.

Least Squares.—The subject of least squares should be required of all students in civil engineering, and probably of those in mining engineering, in the first half of the junior year. Differential equations should be offered as an elective in the second half year.

The minimum course in least squares should consist of about forty formulas under the following headings: (1) Review of theory of probability; (2) Errors and their probability, probability curve, probability integral; (3) Important processes such as solution of observational equations (equal and unequal weights), average error, percentage error, mean error, approximation formulas, propagation of errors; (4) Applications to civil engineering and physics; (5) Normal equations; (6) Fitting formulas to observations.

Many mathematicians intensely dislike least squares and do not want to teach the course. The sentiment seems to be founded on the fact that the ideas and methods of least squares are foreign to a mathematician's usual mode of thought and standards of judgment. The subject does not belong to the domain of logical mathematics, and it is thus different from the usual course in pure mathematics. It belongs to the domain of experimental mathematics, and when viewed in the right light, is really fascinating.

Differential Equations.—The subject of differential equations should be required of all students in electrical, chemical and mechanical engineering in the second half of the junior year. Least squares should be offered as an elective in the first half of the year. In a three hour course the following ground can be covered: (1) Formation of differential equations; (2) Equations of first order and first degree; (3) Equations of first order but not of first degree; (4) Singular solutions; (5) Applications to physics, mechanics, geometry; (6) Linear equations of second order with constant coefficients, with applications; (7) Linear equations with variable coefficients, with applications; (8) Equations of particular forms; (9) Miscellaneous applications.

THE SENIOR YEAR.

Advanced Courses.—The courses for seniors and graduates should be elective. Almost any advanced course may be offered, although it is well to emphasize those which have some connection with applications. Some of the most important courses are: (1) Determinants and theory of equations; (2) Advanced calculus; (3) Fourier's series; (4) Bessel's functions; (5) Spherical and cylindrical harmonics; (6) Vector analysis; (7) Elliptic functions; (8) Theory of functions; (9) Partial differential equations; (10) Linear differential equations.

Advanced Calculus.—Very few departments of mathematics are large enough to offer all the foregoing courses each year. The best that can be expected is to have one or two each semester. Advanced calculus is, in a sense, one of the most important in the list and is worthy of special mention. The minimum course for engineers should include short chapters of an introductory nature on: (1) Infinite series; (2) Definite integrals; (3) Hyperbolic functions: (4) Elliptic functions, Bessel's functions; (5) Beta and gamma functions; (6) Fourier's series.

Conclusion.—Each branch of engineering demands a somewhat different mathematical training from every other branch. For this reason every text book and every course must necessarily be a compromise. There are on the market some very acceptable books on algebra, trigonometry and the calculus. The pressing need is for equally acceptable books on analytic geometry, least squares and differential equations. There are, indeed, some very good American texts on differential equations, but, with possibly one exception, it does not appear that their authors have made any particular effort to find out which chapters are most useful to engineers, or to provide applications adapted to the needs of engineers, beyond certain standard types well known to all mathematicians.

A GEOMETRIC INTERPRETATION OF THE FUNCTION F IN HYPERBOLIC ORBITS, CORRESPONDING TO THAT OF THE ECCENTRIC ANOMALY E IN ELLIPTIC ORBITS.[1]

By W. O. BEAL, Illinois College.

The Geometric Interpretations of E and M in Elliptic Orbits.

If the elements of an elliptic orbit of a planet are given in the problem of the relative motion of two bodies, subject only to their mutual gravitation, the polar coördinates $r = SP$ (Fig. 1) and $v = \angle ASP$, the radius vector and true anomaly of the planet, can be computed for any time t in the following manner:

Compute the mean daily motion

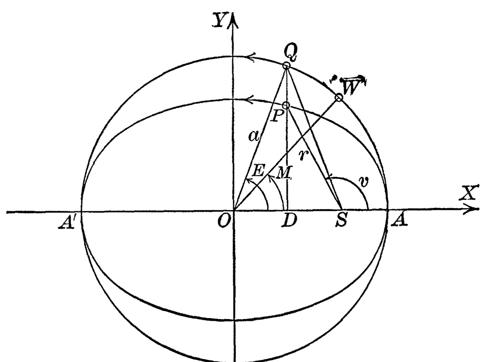

Fig. 1.

$$n = \frac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}},$$

[1] This problem is given by F. R. Moulton in his *Introduction to Celestial Mechanics*, p. 159.